THE
MATHEMATICAL GAZETTE.

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INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1912.

THE MATHEMATICAL ASSOCIATION have undertaken to organise an exhibit, illustrative of Mathematical Education in Great Britain, for the Educational Section of the forthcoming International Congress of Mathematicians, to be held at Cambridge in 1912. It is specially desired to obtain specimens of ordinary mathematical work, both good and average, in various types of schools, as well as models and diagrams. All those willing to co-operate are requested to communicate with the Hon. Secretary of the Exhibition Committee, P. ABBOTT, Esq., Wildwood, West View, Highgate.

THE AIM AND METHODS OF SCHOOL ALGEBRA.[1]

I. THE AIM OF ALGEBRA TEACHING.

§ 1. *Definition of aim.* A distinguished psychologist recently addressed a body of our American colleagues on the teaching of physics. "Your aim in the teaching of physics," he told them, "should be—to teach *physics*." The advice was excellent—especially from one who is apt to be regarded as an intellectual Strong Man, full of esoteric knowledge about the development of mental muscle. Substituting "*mathematics*" for "*physics*" we may well adopt the maxim with all its implications, negative and positive.

§ 2. *Negative implications.* Beginning with the negative implications our amendment of Prof. Baldwin's dictum may be expanded as follows. In the first place, the central purpose in teaching mathematics is *not* to "train the power" of reasoning, of generalising, of "mental accuracy," etc. The fallacies embodied in the persistent heresy of "formal training" have been repeatedly exposed, and need not detain us here. The time should soon come when an educational writer may ignore them. The chemist does not feel bound to begin his text-book by rebutting the phlogiston-theory, nor the geographer to demolish the sophistries of the flat-earthers.

In the second place, its direct usefulness in after-life cannot be given as a sufficient reason for universal instruction in mathematics. As a matter of

[1] These articles contain the substance of the two introductory lectures of a course on *Algebra in the Secondary School* first given at the London Day Training College in the Michaelmas term of 1909 and repeated in 1910.

fact, few of our pupils will need more than a sound knowledge of the multiplication table, the power to check "change" readily, and the ability to keep simple accounts. The argument for carrying the pupil's mathematical learning beyond this point which says that "you never know that it won't some day be useful to you" would lead, if it were taken seriously, to a curriculum of grotesque composition and gigantic proportions. On the other hand, the imposition of an elaborate course of instruction upon a whole school, because a sprinkling of engineers and university scholars[1] may need it in after-life, is a plan obviously unscientific and almost certainly unjust. The universal opinion that every one ought to learn mathematics must have some better justification than this.

§ 3. *Positive implications.* Interpreted positively our dictum supplies this justification by an assertion as simple as it is bold. Mathematics is to be taught, not because the knowledge is necessary for the acquisition of something else, but because it is itself a desirable possession. Mathematics is, like literature and art, a specific form of fine human achievement, one of the cardinal modes in which man has "found himself" and expressed his possibilities. The person who studies it in the right spirit offers himself to one of the great influences that have moulded modern civilisation. For, like St. Paul's Cathedral, the edifice of mathematics is at once its architects' monument and a perennial means of spiritual expansion. In this doctrine is to be found the only ground for teaching mathematics universally that can be held against hostile criticism. It is the ground upon which all the essential subjects of the curriculum must alike make their defence. They represent modes of human activity, intrinsically of such worth, historically of such significance, that every person to be fully educated must have entered into them, at least so far as to acquire a sympathetic acquaintance with their character. *Nil humanum a me alienum.*

§ 4. *Mathematics an activity.* The point of immediate importance here is that mathematics is conceived not as a static body of "truths" but in the dynamic form of an activity. In history mathematics has obviously been an activity, pursued continuously through the generations, and becoming, after the manner of all sustained human activities, more definite in its form, more extensive in its range, more complex in its structure from age to age. To teach mathematics should be to reproduce in the school—as far as the conventional conditions of school life and "the naturally inferior state of boyhood" permit—the essential features of this historic activity.

So much admitted, a two-fold inquiry opens before us. Human activity is always put into movement by mental forces called *motives.* In the first place, then, what are the motives or springs of activity that have generated mathematics in history? In the second place, how far are these motives operative in the immature mind of the schoolboy? For it is evident that if we cannot appeal to the same motives, we cannot really reproduce in him the same activity—we can only force him into a useless simulation of it.

§ 5. *Mathematical motives in history.* Human motives are, proverbially, mixed, and any attempt to separate them for the sake of clearness, runs the risk of falsification. Remembering this danger we may still admit in mathematics and science three sufficiently distinct types of intellectual activity, each dominated by a special motive.[2] In the first type the dominating motive is *simple wonder*—the mind's reaction to and pursuit of

[1] It is difficult to bring girls in at all unless in this category!

[2] These distinctions were first developed by the present writer in an article on Science Teaching in Adamson's *Practice of Instruction* (1906). He described them subsequently in an address on Nature Study given to the L.C.C. Conference of Teachers, Jan. 1907, and in a paper on Epistemological Levels in the *Proc. of the Aristotelian Society*, 1907-8 (cf. Hodson, *Broad Lines in Science Teaching*, 1909, Ch. VI.) They are professedly based on observation, but the philosophical reader will note the parallel with the Hegelian triad, "Thing, Law, System."

the thing that is novel or striking in itself. The fact that if $a^2+b^2=c^2$ the triangle with sides a, b and c will be right-angled is a mathematical example of this kind of thing. To understand the *discovery* (as distinguished from the *demonstration*) of this truth we need suppose no other motive at work than simple wonder. The discovery of the rule of succession of the coefficients in a binomial expansion (when the index is a positive integer) is another example.[1] Most of the sciences begin in this way—"science is the child of wonder," said Plato. The mystical or superstitious attitude towards the properties of numbers which we associate with the Pythagoreans and the mediaeval magicians is largely an expression of this fact. The early history of equations—originating in mathematical conundrums—tells the same tale.

In the second type the attraction of the novel and curious may still be present, but it is displaced from its position as the central motive. That position is occupied by the motive of *utility*. A century or so ago the science of electricity began by delighting our forefathers with pretty discharge phenomena, thrilling "shocks" and weird movements in frogs' legs. But it passed from this stage to produce the lightning conductor and the electric telegraph for the service of man's serious ends. A similar transition occurred in the field of mathematics when, before the dawn of history, some inspired cave-dweller first thought of using pebbles as tallies for business purposes. The documents of the science, from the Rhind papyrus (which tells how the Egyptian priests surveyed the lands left dry by the Nile) down to Prof. Karl Pearson's latest paper on the theory of statistics, give innumerable illustrations of mathematical activity dominated by the motive of utility.

In the third place utility in its turn takes with simple wonder a subordinate place. Reverting for illustration to the science of electricity we have now the stage represented by the work of Clerk Maxwell rather than the work of Wheatstone. The central motive is the *systematising* motive—the impulse to bring some entire province of inquiry under the sway of a body of clearly outlined and closely articulated ideas. Though the consequences of scientific activity on this plane are often both wonderful and useful—like wireless telegraphy, which followed directly from Maxwell's theoretical investigations—yet such consequences are, properly speaking, only bye-products of a process which in itself seeks none of them. Its real aim is indicated by the oft-quoted toast of the *savants* who drank "to the next great scientific discovery, and may it never be of any use to anybody!"

In mathematics Euclid's *Elements of Geometry* affords the best known example of this stage. Euclid's aim was not to make a collection of the most interesting properties of figures, nor to write a handbook for the practical man. Taking a large number of striking or useful geometrical propositions, his aim was to weave them into a logical whole. In other words, he was interested not so directly in presenting the truths themselves as in developing the logical nexus between them, so that geometry as it left his hands might have the highest characteristics of a rational science. Prof.

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[1] Set down a column of units. Construct a second column by adding each figure in the first column to the figure to the right of it, and setting the sum beneath the latter. Construct the third, fourth, etc., columns in the same way. The result is the following table ("Pascal's Numbers"):

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1
		etc.,	etc.			

Chrystal's well-known work on algebra exhibits the same motive at work in the province to which this book belongs. Such a work as Dr. Whitehead and Mr. Bertrand Russell's *Principia Mathematica* shows the rationalising impulse in its present form, freed as entirely as may be from the elements of utility and wonder.[1]

§ 6. *The motives during the school age.* Turning now to the schoolboy, there can be no doubt that he is accessible to the intrinsic wonderfulness of some mathematical facts—for example, that the angle in a semicircle is always a right angle, that the sum of the first n odd numbers is n^2. Most readers can probably remember in themselves a "wonder" attitude towards numbers and their relations—magic squares, the counting into many thousands, "think of a number" riddles, etc.—an attitude which preceded any real appreciation of the *usefulness* of calculation. It is possible that the performances of "calculating boys" are due in part to an exceptional development of this attitude. It is unfortunate that the puritanical traditions of the mathematical curriculum have led to the neglect of the "wonder stage." Its due cultivation would doubtless have a happy effect upon the pupil's subsequent progress.

The interest of the utility motive is, as far as the secondary school is concerned, much better marked, much more persistent and much more important. In science and mathematics it seems to dominate the boy's *natural* intellectual activity, at least up to the age of 15 or 16, and is powerful far beyond that period. That is to say, during the greater part of his school life he is prone to value a scientific or mathematical truth less for its "beauty," and less for the part it plays in a deductive system, than for its usefulness in application. To a schoolboy once known to the writer algebra was a dark and rather unpleasant mystery until inquiries made (out of school) about an extremely simple equation in a handbook on astronomy brought to light the exciting results that could be deduced by algebra from observations of the planetary movements. A year or so later he had taught himself, with the aid of Chambers' *Tables* and *Orr's Circle of the Sciences* (Art. "Nautical Astronomy"), to use seven-figure logarithms and the formulae of spherical trigonometry, so as to check the alleged latitude of his home and the accuracy of the town clock.

Such examples—to which every reader of this article could doubtless contribute—bring out a point of great importance. The mysterious impulse which makes every little boy and girl spend long hours in "make-believe" play is not exhausted during childhood. It persists long afterwards in subtler and more intellectualised forms, which still serve Nature's purpose of making the youth experiment in imagination with life as a preparation for the serious business of living it.[2] The Boy Scout movement has shown us how much intellectual and moral energy this impulse can set in motion in the most unromantic boy. The schoolmaster will do well to capture it for his purposes also.

The rationalising or systematising motive—as defined in § 5—plays an inconspicuous part before the age of 16, and in many boys has never really an independent existence at all. A stimulating teacher may awaken it to a considerable degree of vigour here and there, but his ability would in most cases be better directed to developing the possibilities of the utilitarian motive which exists in every boy. The cause of many of our present discontents is that we persist in regarding as the whole of mathematics what is really only the terminal stage of a process of development—a stage which

[1] This statement is not meant as a prophecy that no useful results will follow from the labours of these mathematicians. History has often shown the precariousness of such predictions. The statement concerns merely the motives to be assumed behind the intellectual activity that produced the book.

[2] Cf. Karl Gross, *The Play of Man*, and McDougall, *Social Psychology*, Ch. IV.

many of our pupils never reach. Here is the psychological principle which justified "the dethronement of Euclid" and other reforms that owe so much to the eloquence, wit, and authority of Prof. Perry.

§ 7. *Applications to algebra.* The foregoing argument affects very definitely the question of the proper aim of algebra teaching. It is possible to form two very different but equally definite conceptions of it. According to the one, algebra is an independent science in the same sense as the geometry of Euclid's Elements is a science. That is, it is a deductive system of truths based upon and developed from a small number of definitions and fundamental laws or axioms.[1] Prof. Chrystal's work has already been quoted as embodying this conception of algebra. The results obtained in such a deductive investigation may have important applications outside its borders, but it is, strictly speaking, no part of the business of the teacher of the science to consider them. If he does so, it is either as a tribute to their *extra-scolaire* importance or "to lend interest" to his own treatment.

According to the other conception, the very existence of algebra and the form which its development takes, are determined by these practical applications. Algebra is a collection of tools to be forged one by one as they are required to deal with problems presented by the physical or social world; a symbolic language specially designed for the exploration and description of phenomena in their numerical aspects.

If the doctrine of § 6 is well founded, these two views do not mutually exclude one another from the curriculum. For each corresponds to a definite type of mathematical activity as we find it in history. But there is the important difference that the second conception implies the utilitarian motive, which is practically universal during the school age, while the former implies the rationalising motive, which has no notable development until the end of the school period, and then only in comparatively few boys and girls.

The second conception should, then, be the teacher's guide in his dealings with the general body of his pupils. The former is relevant chiefly to the needs of the "specialist," and then only after he has traversed with his fellows the ground covered by the utilitarian conception. For, historically and logically, the systematising stage is essentially not a stage of discovery, but consists chiefly in the organisation of results gained on lower planes of intellectual activity. It follows that even the boy of strong rationalising bent will do well not to miss a point of view from which his studies will gain a vitality which the cloistral pursuit of a pure science cannot itself supply.

§ 8. *Interest and discipline.* To say that in teaching we must consider the natural springs of intellectual activity is to enunciate the modern "doctrine of interest."[2] This doctrine is often worst served by those who proclaim themselves its chief friends. It is consequently either feared or despised by many sound teachers who have not taken the trouble to understand it—though they often exemplify its teachings in their own practice. They think that the presence of interest implies *pro tanto* the absence of hard work and "discipline." In reality the presence of interest, properly understood, is the essential condition that hard work shall be profitable and "discipline" actual. It is interest alone that makes boy or man willing "to scorn delights and live laborious days."

For to appeal to interest is to touch the hidden springs of spontaneity; to release the energy that expresses itself in all purposeful activity. The critics of the doctrine constantly confound interest with *pleasure.* They do so because there still lurks in them either the receptacle-for-information view of teaching or the belief that education forms a mind by acting on it from without as the blacksmith moulds and tempers the iron. They think

[1] The laws of commutation, distribution, etc.

[2] Prof. Adams' "Herbartian Psychology" is an account of the doctrine of interest, which has already become an educational classic.

that in some mysterious way either the knowledge or the "discipline" does the pupil good, and that "interest" must be mixed with them as jam is mixed with the powder—and for the same reason.

The doctrine that school mathematics is to be an activity reproducing the *essential* features of historic mathematics is obviously not a "soft" doctrine. For hard and persistent effort as well as exact thinking are certainly among the essential features. But they are characteristics which are not imposed externally; they are the natural marks of intellectual activity when it pursues an end genuinely desired. It would be unreasonable to expect complete parallelism here between the boy and the mathematician. The mathematician selects his own goal and seeks it how he pleases; the boy much accept the choice of his teacher and must work under minutely prescribed conditions of time and place. Apart from his immaturity these differences imply that the internal forces must be supplemented by external influences. It remains true, however, that the boy's efforts are valuable to him only in proportion as they are spontaneous—that is, are the expression of a genuine impulse awakened by an object that really appeals to him.

Misunderstanding on these matters is fatally easy. At the risk of tedium it must be explained that spontaneity does not necessarily mean invention. I show spontaneity in reciting one of Wordsworth's sonnets or in playing a *nocturne* by Chopin if only I "put myself into it." That is, I can use the words of the poet or the musician's melodies as a means of *self*-expression. This observation has an important bearing on the question of "mental discipline." The discipline involved in learning mathematics is, ultimately, the boy's attempt to improve his powers by assimilating the methods of the great performers. It is the 'prentice's striving after the touch of the master-craftsman. In due measure this "formal" element must always be present in an activity which seeks to reproduce actual mathematics. But it must be present as an expression of spontaneity. The purposeless manipulation of symbols has no claim to disciplinary value.

§ 9. *Summary.* Finally it should be urged that the view of algebra teaching here defended is not only sound in itself, but also includes what is sound in views which, as they are commonly held, must be rejected. Thus, on the one hand, it gives due place to the formal or "disciplinary" elements in mathematics; for these are essential features of the historic activity. And, on the other hand, it guarantees the usefulness of the instruction to those who are to enter mathematical professions. For algebra in the utilitarian stage is to be presented not as a gymnastic manipulation of symbols but as an attempt to understand, and so (in due measure) to control the phenomena of nature and society. Taught in this way, the young engineer should not so often find it difficult to see the connexion between the symbolism of his text-book and the behaviour of his machines. The mathematical specialist should not grow into the state of the Senior Wranglers described by Sir J. J. Thomson, who on entering a physical laboratory "are astonished to find that their formulae are true"; while the general body of boys and girls, who will not make direct applications of their mathematical studies in after-life, should cease to regard algebra merely as a "subject" imposed upon them by the caprice of teachers or examiners, and should gain some appreciation of the part it plays in forwarding important human concerns. T. P. NUNN.

THE THEORY OF ORDER, AS DEFINED BY BOUNDARIES.

III. UNIQUE COLLATIONS.

So much for preliminaries. We come now to the formulation of a postulate which may be compared to Euclid's "Let it be granted that

a straight line can be drawn from any point to any other point"; a postulate, by the way, whose significance has been much underrated by later geometricians. To explain fully the closeness of the analogy would involve the discussion of the cataloguing of groups higher than the first, for which I have not space here, but I may indicate its general nature, to those familiar with the subject, thus: In "projective" geometry the existence of straight lines is assumed, and from them, by means of the Quadrilateral Construction, we obtain an unique method of determining any number of points in a straight line, with reference to three given points in it. My procedure is the exact converse of this. I assume what corresponds to an unique method of determining any number of points in a straight line, with reference to any three of them, and from that I deduce a method of drawing other straight lines, that is, if I am given a plane to draw them in. For this is worth noting: The old method of determining points in a line requires a plane in which to do your Quadrilateral Construction. Hilbert has shown that even that is not enough, and that in order to be able to demonstrate certain of the theorems of projective geometry (notably Desargues' Theorem) from what he lays down as the Axioms of Order, without his Axioms of Congruence, not only a plane, but a three-dimensional space is required.

So far, therefore, the comparison is not to the disadvantage of my methods. But before I have done I shall carry the matter a stage further. The postulate which I assume at this stage, the analogue of Euclid's first postulate, is that we have some unique method of determining a collation in the catalogue, if the collations of any three pairs of units are given, or that if we are given the new names of any three units, we can uniquely determine those of all the rest. Later on I shall show that this can actually be done; and that by pure logic, without any objective, or even conceptual reference to Space.

We have seen that we are free arbitrarily to collate any three units in a group of the first order with any other three, or in a collation in the catalogue, arbitrarily to re-name any three units with any three names; and that it is only when we attempt to collate, or to re-name more than three units that the order of the units in the group imposes any restrictions upon our freedom of choice. But we are not bound to avail ourselves of our full freedom, we may specially consider a restricted class of collations which shall be completely determined by naming three pairs of collated units, or three new names in the places of three old ones. Such restricted collations I call *unique* collations, a term which may be implicitly defined by the two formal assertions—

(*a*) In an *unique* collation, or re-naming, the collation, or new name, of every unit of a group of the first order may be determined from the collation, or new name, of any three of them.

(*b*) The resultant of two *unique* collations is itself an *unique* collation; that is to say, if we represent a number of successive collations by a scheme of a number of lines, each being uniquely collated with the next, then the collation of the first and third, or generally of any two lines of the scheme, will be unique.

Now let us determine an unique collation by arbitrarily collating three names, say C_1, K_1, and M_1, with three others; which, to avoid implications as to their order, I will represent by small letters, say, p_2, q_2, and r_2, respectively.

If p, q, and r represent three different names, we may now make another unique collation, collating p_2, q_2, r_2, with C_3, K_3, M_3, respectively; and again we may collate C_3, K_3, M_3, with p_4, q_4, r_4, uniquely.

Then by clause (*b*) of the definition it follows that the collations (1)-(3) and (1)-(4) and (2)-(4) are all unique.

But the collations (1)-(2) and (1)-(4) may both be determined by collating C, K, M, with p, q, r, respectively, and therefore by clause (a) they must be one and the same collation; that is, in collation (2)-(4) every name is collated with itself. And as such a collation is determined by collating any three units, it follows that:

(12) *If three units are collated identically, then every unit is collated identically*; and this particular unique collation may be called *The identical collation*.

Now in (1) and (3) we have C, K, M, each collated identically. Hence this also is the identical collation.

(13) It follows that: *If a collation* (1)-(2) *is unique, so is the collation* (2)-(1).

It may be shown that all unique collations are either positive, negative, or zero collations, that is, that the only discontinuous unique collations are those in which one unit in each group is collated with all the units of the other group; in the case of collations in the catalogue, zero unique collations are, or are not, reciprocal, according as it is the same, or two different units which are so collated.

We may classify unique collations in the following different ways—

Firstly, they may be positive, negative, or zero collations.

Secondly, they may be reciprocal collations, or not.

Thirdly, we can make a classification by the number of units which are collated identically, or we may say, the number of names unchanged by the collation. In this classification we shall have—

(1) More than two units collated identically.
There is, however, only one collation of this class, namely, The Identical collation. It is obviously a positive collation. (Class 3.)

(2) Two, and only two, units collated identically.
All negative collations come under this head. (Class 2a.)
There are also positive collations. (Class 2b.)
Zero collations, if not reciprocal, are all in this class also. (Class 2c.)

(3) One, and only one, unit is collated identically.
These may be positive. (Class 1a.)
Or zero, if reciprocal. (Class 1b.)

(4) No unit is collated identically.
These are all positive. (Class 0.)

(14) The proposition which I am about to prove, viz.: *If in any unique collation any one pair of units is collated reciprocally, then every pair is collated reciprocally, and it may be called a reciprocal collation*, is a very important one, and for this reason alone it would be worth while giving the proof at some length. I do so also to illustrate the methods of reasoning, and the use of schemes; giving first a proof in ordinary language, and then the same by means of a scheme.

Let us consider any four names in the catalogue, let us say C, F, L, and Q. Of these two pairs, (CL) and (FQ) separate one another, while the other two pairs of pairs do not. We have then three cases to consider:
(i) we may obtain a negative collation by collating reciprocally C and F;
(ii) we may obtain a positive collation by collating reciprocally C and L;
(iii) we may obtain a zero collation. I will take these cases separately.

(i) Let us collate C and F reciprocally, and L_1 with Q_2. We may pass the collation under review so that from C_1 we pass to Q_1 and L_1 before reaching F_1. Therefore we also pass from F_2 to Q_2 before reaching C_2. The collation is a negative one, and there must be two names, which we may represent by h and k, which are collated identically.

We might therefore define the collation sufficiently by saying that

h_1, C_1, k_1 are collated respectively with h_2, C_2, k_2, and from this it would follow also that C_1 was collated with F_2, and L_1 with Q_2.

Now let us repeat the collation, collating h_2, F_2, k_2, with h_3, C_3, k_3, respectively. It must follow, as before, that C_2 is collated with F_3, and L_2 with Q_3.

And now, if we pass in review all three names at once, we see that h_1, F_1, k_1 are collated respectively with h_3, F_3, k_3, and since the collations are unique, the collation (1)-(3) is the identical collation. Therefore Q_1 and Q_3 are collated together; and since L was collated with Q, it is collated with Q also. That is, Q and L are collated reciprocally by (1)-(2), as well as C and F. And what is true of Q and L is true of every other collated pair of units.

This argument may be exhibited by the following scheme:

$$\left.\begin{array}{lccccccc} (1) & C & h & F & L & k & (Q) \\ (2) & F & (h) & C & Q & (k) & L \\ (3) & C & h & (F) & (L) & k & (Q) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

In line (1) the letters are written in any order, though it looks better to write them in their real order if we know it. Q, however, is not written in at once, as the brackets show its collation with line (2) will be determined later. In line (2) we write F, C, and Q under C, F, and Q in line (1); h and k represent identically collated units, therefore we write h under h, and k under k; L_2 is written in anywhere. In line (3) we write h, C, k under h, F, k, respectively in line (2). Then we can copy the collation (1)-(2), and so write in (F_3), (Q_3) under C and L in line (2). And since the collation (1)-(3) is seen to be identical, we can write in (Q) in line (1) over L in line (2), and (Q) in line (3). It is now obvious that the collation of Q and L in lines (1) and (2) is reciprocal.

Similarly, if we collate F and L reciprocally, the collation being also negative, if C_1 is collated with Q_2, then Q_1 will be collated with C_2. But if we collate C and L reciprocally, the collation is positive, and the above argument does not apply. The proof in this case is as follows:

(ii) Let us pass in review three names at once. When the first is C let the second be F, and the third L. When the first is F let the second be C, and the third Q. When the first is L let the second be Q, and the third C. That is, the collation (1)-(2) collates C and F reciprocally, and L_1 with Q_2: consequently also Q_1 with L_2. The collation (2)-(3) collates C and Q reciprocally, and F_1 with L_2: consequently also L_1 with F_2.

It follows that the collation (1)-(3) collates C and L reciprocally, and also F and Q reciprocally; and since this is a possible unique collation, it is the only one which collates C and L reciprocally, and F_1 with Q_3.

This argument is illustrated by the following scheme:

$$\left.\begin{array}{lccccccc} (1) & C & h & F & & L & k & Q & \\ (2) & F & (h) & C & r & Q & (k) & (L) & s \\ (3) & L & & Q & (r) & C & & (F) & (s) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Under C, F, L, in line (1) we write F, C, Q, in line (2), and L, Q, C, in line (3). The identically collated names represented by h, k in lines (1) and (2), and by r, s in lines (2) and (3), indicate that the collations (1)-(2) and (2)-(3) are both negative. Consequently, we may write in (L) in line (2), and (F) in line (3), under Q in line (1). On inspection we see that the collation (1)-(3) is reciprocal.

(iii) There remains the case of a zero collation. If in such a collation two units are collated reciprocally, say C and L, one of them, say C_1, is collated also with all the other names in the catalogue, amongst them therefore with C_2. But by hypothesis, C_2 was collated with L_1. Therefore C_2 is

collated discontinuously, and with every other unit in (1). Every collation is therefore reciprocal.

We may therefore determine a reciprocal unique collation by naming any two pairs of units which are to be reciprocally collated. Such a change in the catalogue may be called an *inversion*; we may be said to have *inverted* the catalogue with respect to the two pairs. I find it convenient further to distinguish between positive inversions, which I call *obversions*, and negative ones, *reversions*.

We may also determine inversions in other ways, which differ according as they are negative, zero, or positive. These I proceed to describe.

(15) *A reversion may be determined by naming its two self-conjugate units.*

For though only two units have been named, which at first sight would seem insufficient to determine any unique collation, we are in reality given an additional datum, namely, that the collation is to be a reciprocal one. To show that this is enough, consider the scheme—

$$\left.\begin{array}{llllll} (1) & C & h & (x) & k \\ (2) & (x) & h & C & k \\ (3) & (x) & k & C & h \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d)$$

In lines (1) and (2) we represent a reciprocal collation in which two names, h and k, are identically collated. But, only two given collations, viz.: h with h and k with k, are shown in the scheme, since (x) has not yet been determined (as is indicated by the brackets), and we do not therefore appear to have the means of determining an unique collation. But, to find x, let us add a third line to the scheme, by inverting (1) with respect to the pairs (Cx) and (hk). We see now that in (2)-(3) we have three given collations, viz.: h with k, C with C, and k with h. Thus the collation (2)-(3) is uniquely determined. And we may go back and begin over again, by writing lines (2) and (3) of the scheme down first. They represent a reciprocal collation, since h, k are reciprocally collated. And since C is identically collated (and is not collated with either h or k), it is a negative collation, and therefore some other unit, separated from C by (hk), is also identically collated. This unit, represented by x in the scheme, is therefore uniquely determined; and so we are in a position uniquely to determine the collation (1)-(2).

The case of an obversion is, however, not quite so simple, for in such a collation no one unit or pair of conjugate units is marked out in any way from any other.

Consider again for a moment the above scheme (d). Lines (1)-(2) represent a reversion with respect to (hk), in which (Cx) is a conjugate pair. Lines (2)-(3) represent also a reversion, with respect to (Cx), one of the conjugate pairs of the former reversion. And in this case (hk) is a conjugate pair. Thus it follows that:

(16) *If we revert with respect to a conjugate pair of a reversion, we interchange its self-conjugate units, which form therefore a conjugate pair of the second reversion.*

Let us see what happens if we revert with respect to a conjugate pair of an obversion. The conjugate pairs of the obversion all separate one another, while no two of the conjugate pairs of the reversion do so. Consequently there cannot be more than one conjugate pair common to both. It can be shown that there always is one, thus: if we pass in review the conjugate pairs of the reversion starting from one of the self-conjugate units, h, to the other, k; initially the conjugate pair under review, say (xx'), separates h from the conjugate of x in the obversion x'', for initially x'' is at k. But finally x'' is at h, and therefore it passes the boundary (xx'), and as x'' can never be x, it must have

been x'. Hence in a given obversion any given unit determines, first a second unit, its conjugate, and secondly a conjugate pair, which belongs also to the reversion with respect to the first-named pair of units. A pair of pairs of units so situated with respect to an obversion may be called a *quartette* of the obversion, and we may say that:

(17) *In an obversion each unit determines a quartette.* Since a quartette consists of two pairs of units which intersect one another, it is obvious that, conversely, a quartette is enough to determine an obversion.

(18) We see now, however, that: *An obversion may be determined by naming a conjugate pair and one more unit of a quartette,* for the fourth can be determined by reverting the third with respect to the other two. It will be noted that this does not amount to as much as naming three collations; for the only collations named are the two reciprocal collations of the conjugate pair.

In the last case, that of the zero inversion, it is obvious that to name the one self-conjugate unit is all that is needed to determine the collation. But the significance of this fact, or indeed of zero collations at all, depends, in a manner familiar to mathematicians, entirely on their being *limiting cases* of positive or negative collations. In order to understand this, let us consider the reciprocal collations determined by the conjugate pairs (CF) and (Lq), where q represents a name, initially Q, which we may pass in review round the catalogue. We thus pass in review a group of inversions, and collate them with the letters of the catalogue. They form, therefore, an uniform group of the first order. Initially the inversion is a reversion, for (CF) and (LQ) do not separate one another. There will, therefore, be two self-conjugate units, say h, k; of which one, say k, is separated by (Lq) from all the other named units. If q passes to L, as it reaches L both it and k come to mean L, which is a self-conjugate unit. As q passes on, (Lq) again separates k from the other named units. Now the boundary (hk) constantly divides the whole catalogue into two parts, one contains F and L, the other C and q, and the conjugates of all the units in the other part. Hence, when q comes to mean C, and therefore we have a zero collation, C, and C alone, is the conjugate of all the units in the other part. Therefore the boundary (hk) must consist of the two units contiguous to C, *i.e.* h and k are no longer distinguishable from C, which, besides being conjugate to all the other units, is both self-conjugate units rolled into one. As soon as q has passed C, the inversion becomes an obversion, and continues so till it reaches F, when it again passes through the limiting case of a zero inversion, and reappears as a reversion.

(19) It is obvious that two successive negative collations must effect a positive one. We have already seen this in scheme (c). In particular: *If in an obversion, we revert twice successively with respect to the conjugate pairs of a quartette, we effect the obversion itself.* For consider the following scheme, where the conjugate of any letter with respect to the obversion is represented by the letter with a double accent.

$$\left.\begin{array}{llllllll} (1) & (a) & h & (b) & k & (a'') & k'' & (b'') & k'' \\ (2) & (b) & h & (a) & k'' & (b'') & k'' & (a'') & k \\ (3) & (a'') & h'' & (b'') & k'' & (a) & h & (b) & k \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (e)$$

Here (hh'') and (kk'') form a quartette of the obversion, (1)-(2) is a reversion with respect to (hh''), and (2)-(3) one with respect to (kk''). And evidently (1)-(3) is the obversion itself. But note also that if a represents any name, which reverts into b with respect to (hh''), then the unit collated in the third line with a and b must be called a'', in virtue of the collation (1)-(3), which is the obversion. We may

similarly write a in line (3) under a'' in line (1), and so on, completing line (3). As to line (2), since (1)-(2) is reciprocal, and we have b under a, we have also a under b, and similarly with the other letters, between brackets.

(20) Looking now again at the first two lines of the above scheme, we see that: *Reversion with respect to a conjugate pair of any inversion does not alter the conjugate relation.* For example, reversion with respect to (hh'') changes a into b, but it also changes the conjugate of a in the obversion, a'', into b'', its conjugate in the same inversion. In the case of a negative inversion, a reversion with respect to one of its conjugate pairs merely interchanges the self-conjugate units, in the case of a zero inversion it does not alter the self-conjugate unit, so that in these cases also the conjugate relation is unaltered.

It is worth while adding the enunciations of certain theorems about successive reversions, though, as they do not lie in the direct line of my argument, I shall only add the proof of one of them, which happens to be of special interest.

(21) *Any positive unique collation may be effected by successive reversions with respect to two pairs of units.*

If the pairs do not separate one another, the collation is of class 0.

If the pairs have a common unit, it is of class 1a.

If they separate one another, it is of class 2b.

If they have two common units, it is of class 3 (the identical collation).

The proof of the first proposition is as follows:

In the scheme

$$\left.\begin{array}{llllll} (1) & C & x & F & y & L & z \\ (2) & x & & y & & z & \\ (3) & y & F & x & C & & \\ (4) & & & z & L & y & F \\ (5) & z & L & & & x & C \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f)$$

If we choose any three catalogue names whatever, say C, F, and L, and collate them with any three other names, represented by x, y, and z, then (1)-(2) may represent any unique collation whatever. Invert (1) with respect to (Cy), (Fx) to make (3), with respect to (Fz), (Ly) to make (4), and with respect to (Lx), (Cz) to make (5).

It is evident by inspection that the collations of line (2) with each of the subsequent lines are reciprocal. Thus any unique collation, (1)-(3), may be effected by two successive reciprocal collations; *e.g.* (1)-(2) and (3)-(2). It remains to be proved that if (1)-(2) is positive, the two reciprocal collations may be so chosen as to be both negative.

It will be sufficient to prove that one of them may be chosen negative; since the other must in that case be the same, in order to result in a positive collation. I shall therefore show that at least one of the collations of line (1) with (3), (4), or (5) must be negative.

The condition that the collation (1)-(3) should be positive is that (Cy) should separate (Fx). But, the collation (1)-(2) being positive (see Theor. 7), this is also the condition that in this collation the collated pairs (C_1x_2) and (F_1y_2) should be separated by two identically collated pairs, (p_1p_2) and (q_1q_2), say. Similarly, if the collation (1)-(4) is positive, then (p_1p_2) and (q_1q_2) also separate $(F_1y_2)(L_1z_2)$. But if both these conditions hold, we cannot possibly also have (p_1p_2) and (q_1q_2) separating $(L_1z_2)(C_1x_2)$, and therefore (Cz) cannot separate (Lx), and the collation (1)-(5) must be negative. That is, the positive collation (1)-(2) may be effected by the two reversions (1)-(5) and (5)-(2).

E. T. DIXON.

(To be continued.)

THE "MERANER LEHRPLAN."[1]
MATHEMATICAL CURRICULUM FOR GYMNASIA.

A. LOWER STAGE.

SEXTA. VI.

THE fundamental rules with whole numbers, concrete and abstract, within a narrow range. German measures, weights and coinage. Exercises in the decimal system and in the simplest decimal calculation as a preparation for Fractions.

QUINTA. V.

Calculation. Continued exercises in calculation with concrete decimal numbers with an extension of the range of the measures employed (also foreign weights and coinage), measures of length of different kinds (out of doors); simplest problems of surface- and volume-calculation laying stress on the connection between volume and weight. (In such calculations a rough estimate of the magnitude of the result is always to be made). Numerical factors. Vulgar fractions (first as concrete numbers).

Preliminary Geometry. Introduction to the fundamental ideas of Space as it may be observed, in such a way however that space appears chiefly as involving plane properties. Dimensions, surfaces lines, points, explained first in relation to immediate objects and illustrated from widely different bodies. Plane figures first as part of the boundaries of bodies, then as independent forms in which the ideas of direction, angle, parallelism, symmetry are to be brought out. Exercise in the use of ruler and compass, constant drawing and measurement.

QUARTA. IV.

Calculation. Decimal fractions, contracted methods (in simplest examples). Rule of Three avoiding all exaggeration of stereotyped forms. Problems from commercial life especially simple cases of Percentage (Interest, Discount). Preparation for Algebra teaching by repetition of suitable problems previously solved, using letters in place of definite numbers. Meaning of formulae and their evaluation by substitution of special values. Connection between rules for mental arithmetic with the rules for brackets.

Geometry. Properties of straight lines, angles and triangles; variation of figures in shape and size; dependence of the parts of a triangle one upon the other; transitional cases (right angled triangles, isosceles triangles, equilateral). Simplest properties of parallelogram deduced from the construction of figures.

UNTERTERTIA. III. B.

Arithmetic and Algebra. Systematic co-ordination of the fundamenta rules of Arithmetic by formulae. Conception of relativity developed in practical examples and illustrated by series of natural numbers indefinitely produced in both directions. Computation rules for relative magnitudes. Continuation of exercises in evaluation of formulae, introducing negative quantities and continually insisting on the functional character of the resulting variations in the quantities. Application to pure and complex equations of the first degree with one unknown. Difference between identities and equations.

Geometry. Extension of the parallelogram properties. The trapezium. Fundamental properties of the circle. Consideration of the effect produced on the whole figure by alterations in the length and breadth of various parts. Constructions closely connected with the Curriculum omitting all problems soluble only by artifice.

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[1] Translated for the *Gazette* by Mr. E. A. Price, M.A., Royal Naval College, Osborne.

OBERTERTIA. III. A.

Arithmetic and Algebra. Completion and extension of Algebra, particularly the expansion of polynomials. Simplest propositions in proportion. Pure and complex equations of the first degree with one or more unknowns. Dependence of an expression upon one of its own variables. Graphic representation of simple linear functions and the use of this method in the solution of equations.

Geometry. Comparison of areas and their calculation introducing figures with more complex rectilinear boundaries. Approximate calculation of curvilinear areas. Revision of Geometrical calculations dealt with in Quinta. Problems as in Untertertia.

UNTERSEKUNDA.

Arithmetic and Algebra. Powers and Roots. Pure and complex equations of the second degree with one unknown. Connection between coefficients and roots. Consideration of the variation of a quadratic expression in one variable due to the changes in that variable with graphical illustration. Solution of second degree problems with one unknown by the intersection of straight line and parabola. Consideration of graphical representation as a means of visualizing relations discovered empirically.

Geometry. Similarity, with special stress on similar situation. Proportion in the circle. Calculation of approximate values for circumference and area of the circle by approaching limiting values of inscribed and circumscribed polygons. Exhaustive investigation of the mutual relation between length of side and size of angle in the triangle, especially the right-angled triangle. Construction and verification of Tables for this relation (as preparation for Trigonometry). Practical problems arising therefrom (data with plane table).

B. OBERSTUFE.

OBERSECUNDA. II. A.

Arithmetic and Algebra. Extension of the index idea; conception of the power as exponential magnitude; notion and use of logarithms. Arithmetical Progression (of the 1st order) and Geometrical Progression; use of the latter in compound interest and annuities (in the simplest examples from actual life). Graphic representation of the mutual dependence of a number and its logarithm. Slide Rule. Solution of quadratic equations with two unknowns, both by calculation and by graphic representation.

Geometry—Trigonometry. Deduced from constructive plane geometry. Use in practical problems of measurement of triangles and quadrilaterals. Concrete expressions of the mutual relation between changes of angle and changes of trigonometrical function by the formulae of Trigonometry; graphical representation of this Treatment of suitable problems in several different ways, by construction and with the help of calculation. Investigation of harmonic relations and the principles of modern geometry as the conclusion of plane geometry.

UNTERPRIMA.

Arithmetic, etc. Résumé of functions already dealt with, having regard to their rise and fall over their whole course (with the possible introduction of the ideas of differential and integral calculus), making use of numerous examples from Geometry and Physics, especially Mechanics. Simplest propositions of the theory of combinations with some exercises.

Geometry. Solid Geometry with reference to the most important elements of projection. Exercises in drawing of Solids. Simplest propositions of Spherical Trigonometry. Mathematical Geography, including theory of Map-projection.

OBERPRIMA.

1. Conic Sections treated both analytically and synthetically with applications to the Elements of Astronomy.
2. Revision work taken from the whole province of School Mathematics, and, where possible, harder incidental problems which have to be solved by calculation and drawing.
3. Retrospective work introducing historical and philosophical points of view.

REVIEWS.

Life and Scientific Work of P. G. Tait. By C. G. KNOTT. 10s. 6d. net. 1911. (Cambridge University Press.)

Seldom in a generation does it fall to the lot of a scientific worker to become a pre-eminent leader in his subject; and not often in the course of the centuries is such a leader also pre-eminent as a clear and original writer, as an unusually effective lecturer, and as an exerter of powerful personal influence. Each of these qualifications by itself would be sufficient to make its possessor a man of distinction. He who possesses them all is truly great as a worker and teacher, great also as a man when his personal aims and influences are all in the direction which he deems to be highest and best. Other men gather the long record of work which he leaves behind, and it is passed on to help the work of other ages. His works do not cease with his life, they follow him. His influence continues to spread too, if in diminished stream, unavoidably and unconsciously, by means of those who have come under the compulsion of his personality. He exerts compulsion, not on his pupils merely, but on his compeers also; and through this his influence redoubles itself. From the record of his life and work, men who never knew him may come under the magic spell and be enrolled in the list of his disciples. Such a man was Peter Guthrie Tait, Natural Philosopher.

The portraits with which this volume is illustrated have been happily chosen. Strength, and genius, and cleared-eyed honesty of purpose, are unmistakeably depicted in the magnificent frontispiece, taken at the time when, after the first issue of the epoch-making Treatise, he was occupied with his great paper on Green's and other allied theorems, and with the beginnings of his thermo-electric investigations. Even in the quaint graduate group at Cambridge one can recognise the characteristic pose of the head, the thoughtfulness of the features, and the keen *enquiring* glance, which were familiar in later years. Critchley's snapshot will recall, to 'veterans' (p. 22) who may be alive, the worker and the room behind the lecture hall, which he used partly as a workroom, partly as an apparatus room. The photograph by Marshall Wane exhibits, more than any other, the finer qualities of his nature and the true kindliness of heart which tempered his strength. In the Peterhouse portrait the artist has caught the compelling spirit of the great teacher as he stood in front of his blackboard initiating his advanced class into the mysteries of mirage. In regarding it we can understand better the meaning of Maxwell's words, "Dynamical theorems have been dragged out of the sanctuary of profound mathematics in which they lay so long enshrined, and have been set to do all kinds of work, easy as well as difficult, throughout the whole range of physical science. The credit of breaking up this monopoly of the great masters of the spell, and making all their charms familiar to our ears as household words, belongs in great measure to Thomson and Tait. The two northern wizards were the first who, without compunction or dread, uttered in their mother tongue the true and proper names of those dynamical concepts which the magicians of old were wont to invoke only by the aid of muttered symbols and inarticulate equations. And now the feeblest amongst us can repeat the words of power and take part in dynamical discussions which but a few years ago we should have left for our betters."

The first chapter of the volume constitutes a memoir of Tait, and forms a fitting introduction to the following chapters, in which special features or examples of his work are treated in detail. A proper appreciation of the spirit and meaning of a man's writings is immensely aided if the reader has known the writer's manner

of speech. Next in effectiveness comes such a knowledge of his character and impelling motives as may be gained from a good biographical sketch. The springs of Tait's views and actions are briefly but clearly depicted in this chapter. The sketching, from compulsion of brevity, is necessarily impressionistic; but the essence of the matter is so vividly drawn that each individual reader cannot fail correctly to visualise for himself the great and strong personality, each taking in his own mental presentation of it just that variation of standpoint which gives the result appealing most to his own powers of appreciation, while every such standpoint, so chosen, cannot fail to give a correct view. The essence being correctly given, each presentation consistent with it must be valid. Here are some of the glimpses given, in the form of quotations, in the brief pages:

"In his loveable simplicity and warmth of heart one sometimes forgot his great gifts of intellect";

"No one could know him without being drawn to him by the warmest ties. . . . He was always so hearty and kindly, so ready to help and so pleased to have his friends around him. We all reverenced his gigantic intellectual power and were proud of all that he did for the advancement of science, but the charm of his buoyant and unselfish nature won our hearts from the very first";

"What the Council (of the Royal Society of Edinburgh) now feel is that a great man has been removed, a man great in intellect and in the power of using it, in clearness of vision and purity of purpose, and therefore great in his influence, always for good, on his fellowmen; they feel that they and many in the Society and beyond it have lost a strong and true friend."

Here is one statement from Kelvin:

"We never agreed to differ, always fought it out. But it was almost as great a pleasure to fight with Tait as to agree with him. His death is a loss to me which cannot, as long as I live, be replaced."

Here are some from Flint:

"Undoubtedly it was a great advantage for our students here that they should have entered the Hall through that portal (Tait's class-room), and received the instruction and come under the influence of one universally recognised to have had not only a genius of the first order for research, but rare gifts as a teacher. He was not one whom his students were ever likely to forget, while many of them must have felt that they owed to him far more than they could estimate or express.

"If you have not learned to be interested in the truths of Natural Philosophy, the fault cannot have been your teacher's, and unless altogether incapable of learning anything, you at least cannot have failed to learn that such a man's mind was immeasurably larger than your own.

"Our deceased friend was a man of strong self-consistent individuality. He was 'himself like unto himself alone.' And he had about him the charm inseparable from such a character. He never lost the freshness of spirit which so soon disappears in the majority of men that it is apt to be deemed distinctive of youth. There was to the last a delightful boyishness of heart in him such as is assuredly a precious thing to possess."

"Tait was a genius, but a genius whose life was ruled by a sense of duty, and it was shown to be so by the vast amount of work he accomplished, which is acknowledged by those who are ablest to judge of its worth, to be of the highest value. He was a genius with an immense capability of doing most difficult work, and he faithfully did it. His life was one of almost continuous labour. He faithfully obeyed the injunction, 'Work while it is called to-day.' And the work which he chose to do was always hard work, work which few could do, work which demands no scattering of one's energies, but the utmost concentration of them. He wasted no portion of his time in trying to keep himself *en evidence* before the world. He willingly left to others what he thought others could do as well or better than himself. But whatever he thought it his duty to undertake he did thoroughly."

"He is among the rare few in a generation of whom the memories live through the centuries."

Lastly, here is another statement by Kelvin, made when he was unveiling the Peterhouse portrait:

"Tait hated emotionalism almost as much as he hated evil, and he did hate

evil with a deadly hatred. His devotion, not only to his comrades and his fellow-workers, but also to older men—such as Andrews and Hamilton—was a remarkable feature of his life. Tait was a most attractive personality, and its attractiveness would be readily understood when he unveiled the portrait. It gave one the idea of a grand man, a man whom it was a privilege to know."

No student who desires to appreciate the deep compelling influence which this doer of good and hater of evil exerted upon the fortunate students who came into direct contact with him should read this chapter lightly. A careful study will make him feel the influence himself, and he too will become a disciple, having realised that the influence of such an one never "drops prone and void as any thoughtless dash," but lasts throughout the ages.

The second chapter is devoted to Tait's work as an experimenter from the early days of his collaboration with Andrews to his latest investigations on impact and compressibility. His idea of a college laboratory, given in his address to the students on the occasion of the first opening of his own, was that it should be a place "where a student may not only repeat and examine from any point of view the ordinary lecture experiments, thereby acquiring for himself an amount of practical information which no mere lecturer can pretend to teach him; but where he may also attempt original work, and possibly even in his student days make some real addition to scientific knowledge. That this is no delusive expectation is proved by the fact that in Glasgow, under circumstances as to accommodation and convenience far more unfavourable than I can now offer, Sir W. Thomson's students have for years being doing excellent work, and have furnished their distinguished teacher with the experimental bases of more than one very remarkable investigation. What has been done under great difficulties in the dingy old buildings in Glasgow, ought to be possible in so much more suitable a place as this." It was not merely possible, it was actually done. Tait's early laboratory was, no doubt, much better than Kelvin's early laboratory; nevertheless, like his, it was a poor half-starved affair in which work went on under conditions which repelled any student but one who was an enthusiast. But Tait drew the enthusiasts around him, and the results followed: thermo-electricity and thermal conductivity, pressure errors of thermometers, physical properties of fresh water and sea water, the laws of impact, the motion of rotating projectiles, etc., contributed to a record of work which can easily rank alongside the output of any other physical laboratory. Tait was an enthusiastic experimenter himself, very reluctant to accept a negative answer, possessed of intuitively correct physical instincts which enabled him to see with instantaneous glance the essentials of a problem, and withal very willing to give the performance of work to any other who was eager and willing. He derived even more pleasure from seeing the neat performance of work by another than he could have got from its elaboration by himself. And, as Dr. Knott says, he "never failed to give full credit to those who helped him in carrying his ideas to fruition." It was easy to understand his own remark, "The discovery of a new law of nature, or even a new experimental fact, or the invention of a novel mathematical method, *no matter who has been the first to reach it*, is an event of an order altogether different from, and higher than, those which are profusely chronicled in the newspapers. It is something true and good for ever." So also it was easy to understand the effectiveness of his teaching, and the reverence of his pupils. Even in their junior days he would give his students glimpses of the big peaks and fire them with the mountaineer's ardour.

He willingly left to others what he thought others could do as well or better than himself, said Flint, no matter who has been the first to reach it, said Tait. A good example occurs in connection with the note to Thomson given on page 92. Tait had thought it worth while to enquire experimentally if (presumed) absolute ether drift past two relatively fixed bodies, one emitting luminous vibrations, the other dispersing them on the grating principle, would affect the apparent wave-length. Not possessing a powerful grating, he "set Piazzi on to try his magnificent Gitter." Piazzi Smythe, on his part, having carried out the experiments to a (then) high degree of accuracy, heads his notes "*Tait* experiments by C. P. S." (*Proc. R.S.E.*, 1902). Tait evidently had not sufficient confidence in the admissibility of the theoretical view to lead him to publish the negative result, though he thought the experiment worth performance.

Chapters iii., iv., and v. deal respectively and specially with Mathematical

Work, Quaternions, and "Thomson and Tait" (T. and T'.), although it is impossible to separate entirely the subject matter of each. Tait's mathematical powers were immense. Had it not been so, his work could not have been performed. Kelvin, in speaking of their joint work said that they "did heavy mathematical work, stone-breaking was not in it." But it may probably be said with accuracy that none of his mathematical work originated otherwise than in connection with physical problems. When he was professor of pure mathematics in Belfast, in his first letter to Rowan Hamilton, he said, "I have all along preferred mixed, to pure, mathematics, and since I left Cambridge, where the former are little attended to, I have been busy at the Theories of Heat, Electricity, etc." This fact can form little or no source of regret to the pure mathematician when he considers the other fact that the contributions of physics to the development of pure mathematics have perhaps been as great as have been those of mathematics to the development of physics. Had Tait kept to pure mathematics as his field of work, the gain to pure mathematics would have been large, but his great work in physics would then have been unperformed, and his immeasurable because unrealisable influence on the work of other physicists would have been unexerted.

Amongst the mathematical notes which abound in his Papers, some deal with problems regarding numbers. He once said that he could have given his whole life to the development of the Theory of Numbers; again he said that he had spent two years in the study of the theory and had always considered them as two years wasted. In making the former statement he spoke as a mathematician, in making the latter he spoke as a physicist. Perhaps the most purely mathematical investigation which he carried out is contained in the great series of papers on Knots, which place his name as a writer on topology on the roll which contains that of Listing; but even that work was suggested by physical considerations connected with the possibilities of vortex atoms.

The chapter on quaternions is of unique interest as exhibiting the relations between Tait and Hamilton, the great founder of the method. Work on the projected *Elements* had ceased partly from the pressure of work on definite integrals, but partly also, as it now appears, from an impression that no one cared or was soon likely to care, about quaternions, when Hamilton was led to recognise in his young disciple one who could bear his own armour and wield his own weapons; and then the great work took form with immense rapidity. This is one of the examples of Tait's influence on his compeers. The work of the two was quite distinct: Hamilton dealt with the subject in its purely mathematical aspect, Tait was concerned primarily with its physical possibilites. Maxwell's knowledge of the leading ideas, and physical value, of the method is well known from his introduction of the notation into his own epoch-making treatise: one of the surprises contained in the volume under review is the exhibition given of Maxwell's powerful command of the processes of the method and his clear appreciation of their value in the development of physics. In replying to Tait's transmission to him of two notes, one quaternionic, the other in ordinary analysis but based on a quaternionic treatment, he says, "O T' your notes have ravished me. . . . But the virtue of the quaternions lies not so much as yet in solving hard questions, as in enabling us to see the meaning of the question and of its solution, instead of setting up the question in x, y, z, sending it to the analytical engine, and when the solution is sent home translating it back from x, y, z, so that it may appear as A, B, C, to the vulgar."

The fundamental difficulty in the way of a beginner in the study of quaternions is well put by Tait in one of his early letters to Hamilton. "About quaternions in general I may remark (as indeed I very frequently feel) that the processes are sometimes *perplexingly easy*—by which I mean that one is often led in a step or two and without (at once) knowing it to the solution of what would be by ordinary methods a work not so much of difficulty as of labour. This however I take it must form one of its great excellencies in the hands of a person very well acquainted with it. A drawback to a beginner, but (as I am gradually being led to perceive) an immense advantage to one well skilled in the analysis, is the enormous variety of transformations of which even the simplest formulae are susceptible. . . ." This is indeed the trouble which the average student feels, and there is no cure for it other than that adopted even by Tait—the cure of application which brings familiarity, and with it, in this case, appreciation. Maxwell

well knew, it appears, that, while "unbelievers are rampant," there is no truth in their assertion that "at the best it must rank with abbreviated notations." To some extent at any rate this may be true of non-quaternionic vector analysis; but it cannot be true of the self-contained system which without adventitious aid is developed from a few fundamental definitions and by means of all but one of the rules of ordinary algebra, while that one has the simplest possible substitute. The essence of Hamilton's system, and the feature which determines its peculiar utility, is involved in the recognition of two distinct and non-conflicting uses of the same symbol, as a vector or as a versor. Hence the associative law becomes applicable, and a workable algebra results. If the needless distinction between the two interpretations of one and the same symbol is to be retained, we must give up the simple self-contained quaternionic algebra and either pick out and use its disconnected parts, as is done in the vector algebras or we must introduce, by means of the ordinary imaginary, a separate versor notation, and so retain the associative principle but introduce non-homogeneity as between even and odd products of vectors—to which the logical conscience may object as strongly as to the mixed addition of scalars and vectors.

The fact that there is no real quarrel, whatever discussion there may be, between quaternionists and vector analysts is shown by their conjunction as members of the International Association for the Promotion of the Study of Quaternions and other Allied Systems. Tait was one of its original members, although he declined the honour of its presidency. The only danger that is likely to arise (has arisen) lies in the multiplication of notations; and in this connection it should be remembered that Hamilton's genius was scarcely less exhibited in his invention of a suitable notation than it was in his formulation of this system. His notation could be largely employed in the other vector analysis with no loss to them, but with the advantage that, however different their idioms, their speeches would be expressed in a common language. A quaternionist would only rejoice if in any case such an idiom were more expressive than his own. He rejoices to some extent in the spread of these analyses because, I think, he recognises in it the compulsion of the Zeitgeist unavoidably bringing about the time which Hamilton predicted to Tait in the words, "*Never mind when.*"

The correspondence given in this chapter—Maxwell to Tait, of which the other side is unfortunately lost, Cayley to Tait and conversely—is of great interest. The difference between the antiquaternionic view and the quaternionic view is well brought out in the latter. It is as broad (and no more than as broad) as the difference between pure mathematics and physics. Much information is given regarding Tait's work on "Nabla" and the Linear Vector Function. The man whose good fortune it may be to come into the inheritance which Tait regarded as being still stored up in the linear vector function will owe his title-deeds to Tait. Perhaps Tait got a glimpse of it himself in that last effort of the gigantic intellect just two days before his death (p. 169).

The story of the initiation and growth of the great Treatise is well told in the chapter on "Thomson and Tait." The initiation was due to Tait, and so also was the compulsion which carried it through. The collaboration was very complete; so that, although definite parts are specifiable as the work of one or other, the work of each being submitted to the judgment of the other, it is impossible to say what influences were effective in the final moulding. Each possessed the whole-hearted trust of the other, and neither thought of self. Tait's self-obliteration appears in two connections. One is the exclusion of quaternions. No one can judge which author was right; whether the introduction of quaternions would have diminished the usefulness of the Treatise, or whether it would have compelled as great a change in the treatment of the mathematics or physics as it did in the treatment of dynamics itself. The other case of effacement occurred in connection with the treatment of attraction. Tait wished that subject to be reserved for the projected volume on properties of matter. It is interesting to note Hamilton's disciple introducing Thomson to the Hamiltonian dynamics. How delightedly he would watch the developments! A good example of the converse action is seen in Thomson's incitement of Tait to enquire into the foundations of kinetic theory.

The entire loss of Tait's side of the Maxwell-Tait correspondence is greatly to be regretted. The nature of the letters can only now be inferred from the

replies, of which some are given. They must have contained rich examples of that bubbling humour and of the apt quotations which Kelvin spoke of as having "brightened their hours of work." This note to Kelvin gives an amusing illustration:—To W. Thomson, 20/10/83. *Dat veniam corvis*:—*e.g.* "34 pounds on the inch" J. & T. Has even *this* made its authors the laughing-stock of Europe? *Vexat censura columbas*:—*e.g.* "3 tons weight per square inch." T', pitched into by T., in the quasi-prophetic terms quoted above. *In memoriam peccati, Haud immemor peccavisti.*

Chapters vi., vii., and viii. contain respectively accounts of Tait's other books, his addresses and reviews, and his popular scientific articles. All of these were characteristic productions. Among the samples given of the latter is his non-mathematical account (published in *Nature*) of his beautiful investigation, experimental and mathematical, into the phenomena of mirage. Another, sampling his papers on the science of golf, is the article on Long Driving from the *Badminton Magazine*. In connection with this subject, and also as giving a vivid picture of the man in the midst of holiday surroundings, a special section on "Tait at St. Andrews," written by J. L. Low, the biographer of his son, Freddie Tait, has been included in chapter i.

In the course of the volume, two or three examples are given of that power of witty versification which Tait possessed in common with Maxwell.

Tait's attitude towards matters of religion is also dealt with. Its general nature is well known to any one who is acquainted with *The Unseen Universe*. His view of those who regard religion and science as mutually incompatible is well put in his article on "Religion and Science," reproduced in chapter vii. from the *Scots Observer*. His own attitude is still more clearly put on p. 36, in an extract from an article, "Does Humanity Demand a New Revelation?" published in the *International Review*. On the next page Dr. Knott gives an instance of Tait's strong disapproval of the flippant use of Biblical quotations, absolutely correct in so far as Tait's part is concerned. But the man who made the quotation objected, as strongly as any one could, to such uses. The silence to which he was reduced was pained silence, not astonished silence; for it was Tait's remark only that revealed to him the peculiar sacredness of the quotation he had made, and which he had for the moment entirely forgotten. I only allude to the matter because of the fact, of which Dr. Knott must certainly be unaware, that the man is still alive and might see the statement. Dr. Knott has done what must be regarded by the vast majority of his readers as a true service in bringing these expressions of Tait's views together. For Tait had, on this subject, the characteristic Scottish reticence which is born of a reverent attitude of mind toward things sacred. He shrank from open expression regarding them. Here, for example, is a striking thought, written marginally, on a fly-leaf, around some mathematical notes, in which he suddenly translates the remainder of a sentence into French:—"Surely a *thought* or a *feeling* is of higher class than a particle of matter! Can *we* originate the one when we can't make or unmake the other? Cometh from the Père des lumières avec qui il n'y a pas de changement ni l'ombre du versement."

The volume forms a fitting companion to the two volumes of Scientific Papers published, during Tait's life, by the Cambridge University Press. Every student of physics who reverences the work and the personality of a great leader should have it on his shelves. The price is so small that it can hardly exceed the cost of production. I can give to the author no higher praise than that involved in the expression of my belief that his task could not have been better performed.

W. PEDDIE.

A New Geometry. By W. M. BAKER, M.A., and A. H. BOURNE, M.A. (G. Bell & Sons.) 2s. 6d.

Experimental and Theoretical Course of Geometry. By A. T. WARREN, M.A. 4th edition revised. (Clarendon Press.) 2s. 6d.

A First Book of Geometry. By J. V. H. COATES, B.Sc. (Lond.) (First Books of Science Series.) (Macmillan & Co.) 1s. 6d.

In all the multitudes of geometries which have been written in accordance with the Report of the Committee of the Mathematical Association to take the place of Euclid for school purposes the work has been separable into three main divisions. First comes the preliminary portion devised for the purpose of

making the beginner familiar with geometrical ideas and with simple constructions in which no formal methods of proof are given, but appeal is made to instinct and general common notions. Then comes the formal work divisible into the two parallel courses of Theorems and Problems.

The preliminary portion is treated very differently by different authors, and probably no one treatment can be considered as distinctly superior to any other for all classes of pupils and all schools.

It is here particularly that every teacher must exercise his own judgment, utilizing such portions of the author's exposition as he thinks best for his class. In many cases he will probably prefer to strike out a line of his own. The object aimed at is to lead on the pupil from familiar things to the abstract ideas required for a firm foundation on which to build the subsequent theory. There is a danger of this part of the work being made too elaborate and being taken too far, leading the pupil to be content with somewhat slipshod arguments, or bewildering him with a multitude of half digested ideas. This danger is alluded to in the circular issued by the Board of Education in 1909 in which they say: "A few teachers have gone to an extreme in thinking that practical work is everything, and have sacrificed the training in rigid deduction which is an essential element in school geometry."

It seems to me that Mr. Warren's book errs in this respect, as 113 pages are devoted to the experimental course, and the constructions given in this course are not again given formally in the systematic course which follows. The two parts can, however, be read together, references to the appropriate parts of the experimental course being attached to some of the propositions, and the fact that the book has reached a fourth edition shows that its scheme has found considerable favour.

Messrs. Baker and Bourne have rather fallen into the other extreme. They have an interesting short preliminary course consisting of a series of questions carefully chosen, and then plunge into a rigid deductive course with little or no discussion of new ideas as they arise. For example, Playfair's axiom is inserted at the foot of a page with no reason given for the necessity of its introduction.

By far the most stimulating preliminary course (where conditions enable it to be carried out) is given in Mr. Coates' little book, in which the work indicated is practical surveying with simple apparatus, work which would be of the utmost value to pupils who can actually be taken through it in field and classroom by a teacher possessing enthusiasm, practical ability, and power of organization. Mr. Coates showed his apparatus, made by the boys themselves, and explained his method of instruction, at a recent meeting of the Londo Branch of the Association.

In the theoretical course there is a danger of the early theorems, especially those treating of parallels, which are the scientific foundations of the subject, being not quite sound. In this respect Messrs. Baker and Bourne's treatment is the most careful and scientific, though their theorem 4, excellent as it is, proving that the exterior angle of a triangle is greater than either of the interior opposite angles, will probably be found rather difficult to a beginner, who will also, as mentioned above, not readily realize the force of Playfair's axiom without further explanation. Mr. Coates has missed a chance of clinching a very pretty method. He says, by way of definition, that two lines in a plane which make equal angles with a transversal are parallel, and then by help of Playfair's axiom, proves that if two lines are parallel they must make equal angles with a transversal. This statement, written out at full length, reads thus: If two lines in a plane make equal angles with a transversal, they make equal angles with a transversal! which would seem rather self-evident. As a matter of fact, he has proved that if they make equal angles with one transversal, they also make equal angles with every other transversal, which is a very important theorem. It is a great pity that he has not made this clear. Also, curiously enough, he has not shown that lines so defined will never meet, though it would not be difficult to do this.

Mr. Warren's method, by rotation of the transversal, will I think be found very confusing and difficult. Positive and negative rotations are difficult ideas for beginners, and even if the point is grasped it will still be difficult to write out. We have to think not only of an argument being understood, but also

whether it is an easy one to reproduce on paper. It would be good to have a discussion on the best method of giving the theory of parallels to beginners, and this might be done either at the London Branch of the Association, or at a general annual meeting.

Mr. Warren's treatment of equality and inequality of sides and angles of a triangle by folding about the bisector of an angle is very neat, and so is his similar treatment of the congruence of triangles. All are done on one plan, and he has brought the method of folding into great prominence, and with excellent effect. A good many of Mr. Coates' theorems are proved in the same way, but he does not use the method to quite the same extent. Both authors avoid the "reductio ad absurdum" as much as possible, while Messrs. Baker and Bourne approximate more to Euclid's methods, and use the "reductio" proofs much as he does, and they depart from Euclid's procedure only where it is generally agreed that the modern methods are simpler or more practical, *e.g.* Euc. I. 5, and the reduction of a polygon to one with fewer sides, extreme and mean ratio, and a few others, in some of which the method of folding is utilized. The other authors depart much more from Euclid's text: see, for one example, their respective proofs of the theorem of Pythagoras, which are proofs by transposition of parts.

Messrs. Baker and Bourne cover the ground of the parts of Euclid usually read, including the substance of Book XI., and conclude with a chapter on the mensuration of solids.

The chapter on similar plane figures is very complete and includes a good many practical constructions.

A number of valuable exercises are given throughout the book.

The only serious omission seems to be the absence of methods for constructing equivalent parallelograms or triangles on different bases, and the theorem of the equality of the complements of the parallelograms about the diagonal of a parallelogram (on which Euclid founded such constructions) is given only among the examples (p. 50) with nothing to indicate its importance and ancient interest. Also problems and theorems are somewhat intermixed, so that it is not very easy to turn to any required problem. But on the whole the book well covers the course of geometry required for non-specialists in public schools, at any rate below the higher forms, and is a scholarly and well written book. The most serious blemish is their so-called proof that any two sides of a triangle are together greater than the third. It would have been better to boldly put it as an axiom, or as a particular case of the axiom that a straight line is the shortest distance between two points. If it is given as a proposition at all, what is better than Euclid's proof? Again, the solution given on p. 192 (extreme and mean ratio) would hardly be accepted in an examination. There is a misprint in the diagram of prop. 16 (p. 138), a minus sign being omitted in each figure.

Mr. Warren's book covers much the same ground in plane geometry, but does not proceed to any solid geometry. It concludes with a useful set of examination papers, London Matriculation, Locals, and Naval Cadets.

Mr. Coates' book comes to an abrupt end halfway through the properties of a circle, and his constructions are to be found only in the experimental part of the book, mixed up with other matter. It concludes with a useful set of additional exercises and some local exam. papers, and there is an index to facilitate reference. Anyone who has worked through the book will have a good knowledge of elementary geometry up to the point reached, but the necessary transition to another book by those who are going on to higher work would be rather awkward. Perhaps Mr. Coates is intending to write a sequel which will naturally fit on to this excellent first course.

It is usual, nowadays, to take older pupils through a short course in harmonic section, poles and polars, radical axes, inversion, and to deal with the harder tangency problems. None of these is treated in the books now under review.

L. A.

(1) **An Elementary Text-book of Trigonometry.** By R. W. K. EDWARDS. 1911. (Alston Rivers.)

(2) **Elements of Plane Trigonometry.** By DANIEL A. MURRAY. 1911. (Longmans, Green.)

(3) **Elementary Trigonometry.** By W. E. PATERSON. 1911. (Clarendon Press.)

(4) **Elementary Trigonometry.** By F. T. SWANWICK. 1911. (Cambridge University Press.)

Trigonometry may be roughly divided into three sections: (1) Arithmetic, (2) Geometry, and (3) Algebra; the first section dealing with use of tables for solution of triangles and the like, the second with Modern Geometry, the third being Analytical Trigonometry so called. It is generally accepted that for 'Beginners' the first section is sufficient, and this section forms the subject of the four books before us, each book approaching and continuing the matter in a different manner.

(1) This is written on somewhat orthodox lines, with the exception that circular measure is introduced on p. 12, and that a short section on hyperbolic ratios is given at the end. The reason for the early introduction of circular measure is not apparent from the subsequent matter; and the use of the new-fangled notation for hyperbolic ratios is likely to worry a student when he gets on to 'Hobson,' but it has the merit of giving pronounceable names to these ratios. The exercises are generally well thought out and are fairly numerous. The value of the book would be greater if not marred by numerous minor points, such as the following. The explanations of infinity and the continuity of the graph of $y=\tan x$ are not lucid enough. The initial presentment of the six ratios together is hardly to be recommended; whilst the definition of the ratios of angles of any magnitude from the sides of a *right-angled triangle*, without any mention of the sign convention (although MP instead of PM is rightly insisted upon), is illogical, for the triangle does not bear the same relation to angles not acute that it does to acute angles. It is also liable to lead to confusion, as instanced by the proof of '$\sin(A+B)$' on p. 95, where $\frac{NM}{OM}$, $\frac{ON}{OM}$ are stated to be ratios of XOH instead of the reflex angle XOM. The use of Sin^{-1}, etc., for general values is not always adhered to. Page 20 would look far better without the truly awful mnemonic at its foot; the coordinate definition of the ratios avoids such things as this. It is *not* correct to use $\tan\frac{1}{2}A=\sqrt{(s-b)(s-c)/s(s-a)}$ in all cases, for instance, when A is a large obtuse angle; similarly, if B and C are very nearly equal the formula $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\tan\frac{1}{2}(B+C)$, and the above formula for $\tan\frac{1}{2}A$, when A is small, are unsuitable for logarithmic work without a special formula, such as Maskelyne's. It would be far better if more applications to the geometry, which the author affects to despise as a great mental training, were given. We only notice one misprint—page 57, Ex. 18.

(2) It is difficult to find a place for this volume in the curriculum of English schools at the present time. It covers the ground of the old examination syllabus of "Elementary Trigonometry to the Solution of Triangles." In fact, solution of triangles (*i.e.* Arithmetical solution) seems to be the end and aim of the volume; this idea is strengthened by the accurate but practically useless *cardboard* protractor inserted at the end of the book. For a book of this kind the price is too high (3s.).

Some minor points of criticism are:

Grades are defined and said to be used in France, the treatment of circular measure, which is introduced very early, is insufficient for beginners, and the graphs of the functions are badly drawn: the wide leading of the type makes the proofs very sprawly and hard to follow, and the pagination is frequently faulty, the single-line conclusion of a proposition in one case appearing overleaf (p. 98).

(3) The method of this volume is very good. The author recognizes as mistaken the idea that the ratios are fundamentally attached to a triangle and gives the right-angled triangle definitions only when p. 86 is reached and solution of right-angled triangles is begun. For practical examples, the excellent idea is followed of using interesting problems on Navigation as well as applications to Geometry, Mensuration, and Physical formulae. The discussion of '$A+B$' is deferred till the solution of triangles has been treated fully: and then the addition and allied formulae are developed by projection. Unfortunately the proof is made to depend on the formula $\cos(90+A)=-\sin A$, which does not seem to have been proved to hold for all values of A, in fact it is the one formula

of the allied set which has been omitted and left to the student to prove when the rest of the set have been proved for the case when A is acute. This also renders the proof of the product form for $\cos A + \cos B$ non-general. Demoivre's Theorem is made to depend on a *suggested* geometrical interpretation of ι which is not shewn, or stated, to satisfy all the laws of Algebra; accordingly, as it does not seem to be used afterwards, it would be better omitted. Graphs of the Trigonometrical Ratios in general are given far too early. The definition of the grade, especially as it is stated to be obsolete even in France, might well have been left out. It should be stated that "readers of this volume will always understand that $\sin^{-1}a$ will stand for the smallest positive angle that has its sine equal to a"; for it is usually understood to be the smallest acute angle, positive or negative. Infinity is not a number. The notation "$\sin\theta = \theta = \tan\theta$ when θ is small" is not to be recommended; limiting values should always be given as a ratio (or a difference) when possible. The proof of the formulae for $2A$ should have a second figure drawn in which $2A$ is obtuse, or else the formulae for $\sin\frac{A}{2}$, $\sin\frac{A}{2}$, $\tan\frac{A}{2}$ in terms of the sides of a triangle fail when A is an obtuse angle. On page 109, in the midst of projection, there appears a bad misprint, viz. :—$AB = AZ + BZ$.

(4) This volume is, as a whole, altogether excellent, both as regard to matter, style, examples, and printing, though some points call for criticism.

It opens with a discussion of approximations and limits of error, which more properly should find a place in Algebras, but rarely does. Since it is given here, it should however be given completely and accurately. Thus it should be shown that the error in taking $1+\frac{a}{2}$ for $\sqrt{1+a}$ is less than $\frac{a^2}{8}$, the error in taking a/b for A/B where $A = a \pm \alpha$, $B = b \pm \beta$ is less than $(ab + \beta a)/b(b - \beta)$ and so on, and then the relative errors found: also, it should be explicitly stated when the divisor in a contracted division may be curtailed, and when the process in a square-root calculation may be continued as an appropriate division. It is a mistake to use PN/OP for the sine, even if the sense of the lines is understood to be unconsidered. It would be just as intelligible to the student if NP/OP were used, and he would not be worried by the change in definition later, when sense is taken into account.

The values of $\cos 0°$, $\sin 0°$, $\tan 90°$ are well explained from the limit idea; and solution of right-angled triangles has a prominent place, though the author does not fall into the error of overdoing this section.

In Part II. the method of *defining* $\cos(\text{obtuse}) = -\cos(\text{supplementary acute})$, $\sin(\text{obtuse}) = \sin(\text{supplementary acute})$ *for convenience' sake* is used. This method, though not new in Trigonometrical text-books, has not met with the wide adoption it deserves. According to modern ideas of successive small extensions of definitions according to needs, it is quite the correct idea; it is also a fruitful method from the teacher's point of view and one which should be more widely used. The '$A + B$' formulae can then be proved simply, even when A, B, or $A + B$ is obtuse, from the triangle. Thus, in Part III., the author is able to develop without a break all the formulae for angles of any magnitude with a still further extended definition of the ratios. The notation $\sin^{-1}a$, usually slackly treated, is here carefully discussed, and we are led up to a projection proof of $\cos(A + B)$. The part is concluded with a short chapter on triangles and quadrilaterals and connected circles. The hyperbolic ratios are apparently considered to be out of place in an elementary text-book.

The bad feature of the book is the long chapter system employed; in several cases there are more than a dozen pages without an example, followed by some 50 to 100 examples. This would militate against its adoption as a class book, unless in future editions the sets of examples are broken into sections, each section providing the teacher with a small set on the work of, say, three or four pages.

J. M. CHILD.

Theory of Groups of Finite Order. By W. BURNSIDE, M.A., F.R.S. Second edition, 1911. (Cambridge University Press.)

All who value really fine work in pure mathematics will welcome this second edition, and join in congratulating the author on the call for it. There are in it

large additions to what the first edition contained, and the old matter has been a good deal rearranged, but it remains an exposition of *pure* group-theory, not elaborating particular sub-theories except with a view to illustration or representation, and not concerning itself with applications to the Theory of Equations or other branches of mathematics. In the first edition the special theory which at the time of issue was known as that of groups of *Substitutions*—Mr. Burnside now discards the term and speaks of *Permutations*, as is far better, and as he might with advantage have said explicitly—was kept in the background as long as possible, after a mention of its ideas and its notation for use in making conclusions tangible by exemplification. Historically, as a study of groups of operations of a particular kind found to be of importance for a particular purpose, the discussion of the solubility of equations, this had led to the formulation of a body of doctrine about groups of operations of any kind; and the time had come to exhibit this doctrine as abstract, not letting the support of it seem to rest on any concrete notions with which its origin had been associated. Only in virtue of a theorem that every group of finite order can be represented as a group of substitutions (permutations) did the author, later in the book, relax the severity of his exclusion of reasoning by aid of the concrete, to the extent of admitting proofs depending on the possibility of this representation. Other classes of groups which occur in mathematics were but lightly referred to. The theories of such were still only, or in the main, receivers from the abstract theory, and not helpers in constructing it, suspected or unsuspected.

But in the interval since 1897, Mr. Burnside and others, among whom Frobenius is conspicuous, have put one special group-theory on quite a new footing. Then our author said: "It would be difficult to find a result (in the pure theory) that could be most directly obtained by the consideration of groups of linear transformations." Since that time he has been busy attacking and overcoming the difficulty referred to, and at present he writes: "It is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions." His own researches have in fact been so illuminating that the most modest of men may not exclude them. There remains the old sternness of abstention from secondary methods in most of the earlier chapters, where properties of groups capable of clear expression without reference to modes of representation are developed, but later on, not only permutation groups in particular, but the more general groups of linear substitutions—there is a noticeable refraining from the bold use, with Hilton for instance, of the old term substitution groups in its new wide sense—are given the prominence to which they have been shown to be entitled. Herein lies the great new departure of this second edition. The well ordered account of Group-Characteristics and their importance in chapters xv. and xvi. is part of the novelty, and is a notably complete exposition of what has so far been discovered in a field by no means yet exhausted.

The book contains other additions from the fruit of recent research, which are not connected with any new departure.

Mr. Burnside's expression is terse, and at the same time exact even to punctiliousness. All diffuseness he seems to abhor, and of inattention to a minor difficulty while facing a major he is clearly incapable. To miss nothing in his arguments requires the closest of attention, but success in numerous efforts to convince oneself that he says, and says rightly, just all that is necessary in particular connexions, inspires confidence that if there are any misconceptions in the book they must be very few. To criticise the arrangement of the matter in sequences of connected theory, decided on as it has been by one supreme in his mastery of the subject dealt with, would savour of presumption, and is beyond the competence of the writer of the present notice. E. B. E.

Analytical Mechanics, comprising the Kinetics and Statics of Solids and Fluids. By E. H. Barton. Pp. xx + 535. 10s. 6d. net. 1911. (Longmans, Green.)

Prof. Barton's text-book should not, we consider, be attacked by students who have not already an elementary knowledge of the methods of the calculus. As will be seen from the title, it covers a wide ground, the order of treatment being Kinematics, Kinetics, Statics, Hydromechanics, and Elasticity. The book is

so arranged as to be available for the various courses required for the different examinations, and is complete in itself. In the sections dealing with Kinematics, mechanisms and strains are included. References are given throughout to works which may be usefully consulted by the student. A chapter on "The Physical Basis" deals with the contributions of Newton to Mechanics, and with the attitude of Mach, Pearson, and Love towards the principles of the subject. In a text-book of this size more space might have been allotted to the bending of beams. Prof. Barton attaches little value to the worked-out example. But those at the conclusion of his chapters are all carefully selected, and "at the end occur additional examples of a harder or more varied character, some classified, some miscellaneous, also subjects for essay-writing." The total number, many drawn from the degree examinations of the Universities of London and Calcutta, is about eight hundred. From the remarks made by the author upon the use of Atwood's machine the student will hardly gather the absurdity of using it for such a purpose as the determination of "g." The book is singularly free from misprints. We have noted but one, and that is merely the "1" dropped from "log." It is admirably arranged, and is clearly the work of a skilled teacher. The diagrams are good and the volume is well got up.

Text-book of Mechanics. Vol. III. **Mechanics of Materials.** By L. A. Martin, Jr. Pp. xiii+229. $1.50 net. 1911. (Wiley, Chapman, Hall.)

The subject-matter of this volume of Prof. Martin's work on Mechanics deals with what we should call in this country "Strength of Materials." The author claims that "an attempt has been made to produce a book which will encourage the student to think and not to memorise, to do and not simply to accept something already done for him, but still furnishing sufficient material in the way of explanation and example so that he will not become discouraged." We think that Prof. Martin has fairly substantiated this claim, and that this little text-book will prove an adequate introduction to a subject of great importance in the training of the engineer. The book is well arranged, the exercises are carefully graded, the illustrations are ample, and the publisher has been equal to his responsibilities, save in taking the trouble to provide a proper integral sign on p. 188. Occasionally the author is hardly as clear as one might wish. For instance, we find the following utterance rather cryptic: "Although calculus is freely used it is written for beginners who have had some previous training in theoretical mechanics." On p. 76 we are told that when dy/dx is very small then $(dy/dx)^2$ "is negligible when added to unity." We are not convinced that all teachers will agree with the method adopted by the author in his treatment of the deflection of beams. He finds the differential equation of the elastic curve and applies it at once to a series of concrete problems. Many will prefer to integrate the equation twice, and thus obtain a perfectly general expression for the deflection of a beam with *any* system of loading. The solution of any beam-problem then merely presents the one difficulty—that of drawing the B.M. curve for that particular case, the maximum deflection being obtained from the general equations in the usual way.

Systèmes Cinématiques. By L. Crelier. (No. 31, **Scientia,** Gauthier-Villars.)

Dr. Crelier's little monograph of 100 pages is rightly prefaced by a portrait of the late Colonel Mannheim, whose researches, to be found in his *Principes et Developpements de Géométrie Cinématique*, form the *point de départ* of the kinematic method of curve-tracing. The value of the book would have been greatly enhanced had the author prefixed a short sketch of kinematic geometry, for the reader could then have entered without further preliminaries upon its application to the study of plane curves. We have here to do with the relative motion of two planes, the one sliding over the other, which is fixed. In particular cases he investigates the paths of certain points, the envelopes of certain lines, roulettes, etc., and deduces the properties of certain important curves. The six systems studied are generated by: (i) a right angle whose vertex moves on a given line, and one side of which passes through a fixed point (conchoidal); (ii) a right angle, one side of which passes through a fixed point O, while a fixed point on the other side lies on a fixed straight line passing through O (this gives the Kappa system, so-called from the likeness of the figure to the Greek letter κ; the

curves were known to Newton, Barrow, Jean Bernoulli, etc.); (iii) a right angle, one side passing through a fixed point, the other side being of constant length, and its extremity describing a fixed line (strophoidal system); (iv) a straight line turning round a fixed point while a fixed point upon it describes a circle (the circular-conchoidal system, slide crank); (v) the elliptic compass; (vi) slider-crank-chain. The author may be congratulated on the methodical manner in which he has carried out his task. Most of the results are, of course, well known, but here and there one comes across relations between the various curves of the same kinematic family which are to the best of our knowledge due to his own researches. We have found the figures of little use. Many are a mere blur of lines and letters. Fortunately, the reader will not lose by having to construct the limited portions or outlines that are necessary for the thorough comprehension of the separate sections.

B. G. Teubner, 1811-1911. Geschichte der Firma in deren Auftrag herausgeben. Von Friedrich Schulze. 1911. Pp. 520. (Teubner, Leipzig.)

On Feb. 11th the great house of Teubner, *doyen* we may suppose of publishing houses on the continent, completed its centenary. The occasion was, of course, celebrated by great festivities and a gathering of men of letters and science, and in addition to the arrangements appropriate to such an event we have the handsome volume that lies before us. B. G. Teubner was born in 1784 and died in 1856, and here we have a most interesting account of his career, and of the labours of the successors who have made their house the premier firm in Germany. About 100 pages are devoted to mathematical publications. In these later days it is a standing marvel how any publishing firm could hope to reap financial profits on many of the invaluable works presented to the world through the enterprise of Messrs. Teubner, and one is driven to the conclusion that here at any rate is one publishing firm that is animated by something other than the mere desire for gain. The value of the volume is greatly enhanced by the large number of facsimile letters, etc., and the portraits of famous contributors to the various series issued by the house. Nor must we forget to draw attention to the excellent groups of portraits to be found in these pages. We pick at random from those of mathematicians: Gauss, Euler, Jacobi, Schlömilch, Hankel, Harnack, Kronecker, Stolz, Reimann, Grassmann, Clebsch, Lie, Minkowski, Plücker, Steiner, Abel, Lobatschefski, and (the only representative of this country), G. Salmon. Each page is printed in large and beautiful type with an edged border, and the whole tastefully bound in linen. There is no indication that the volume is for sale, so we are unable to inform our readers if they will be able to obtain this fitting tribute to the energy and the high traditions which have always been a distinguishing characteristic of the house of Teubner.

Leçons d'Arithmétique théorique et pratique. By J. TANNERY. Sixth edition. Pp. xi+545. 1911. 7 frcs. (Armand Colin.)

There is nothing in the English language with which we can compare Prof. Tannery's masterly volume on the principles of Arithmetic. It forms part of the complete course for the *Classe de Mathématiques* A, B, for which Prof. Darboux is responsible. Needless to say, it is not a work that is intended to be placed in the hands of beginners, but it is a volume in which the teacher of the elements of the subject will find a veritable storehouse of valuable suggestions. It is notable for the manner in which the author opens up the subject, in which little by little he proceeds from the concrete to the abstract, until the student is at last brought to grips with what we in this country do not associate in our practice with Arithmetic at all. David Hilbert has recently lamented the fact that the number of devotees of the theory of numbers, especially outside Germany, is discreditably small. The elements of that theory should be the common property of all mathematicians. "We value the simplicity of its foundations, the exactness of its conceptions, and the purity of its truths; we extol it as the pattern for the other sciences, as the deepest and most inexhaustible source of all mathematical knowledge, prodigal of incitements to investigation in other departments of mathematics."[1]

[1] Preface to L. W. Reid, *Elements of Theory of Algebraic Numbers.*

If to those who propose to be mathematicians Arithmetic were taught on the lines laid down by Prof. Tannery we can quite believe that Hilbert's Jeremiad might be turned into a Paean, and that, as far as this country is concerned, a reproach would soon be removed from the land that gave birth to an H. J. S. Smith, and in which we have for nearly twenty years been awaiting the appearance of the second volume of our only great work on the subject. However that may be, it is sufficient here to say that under the guidance of our author Class A and B is led through the elements to the law of the reciprocity of quadratic residues, "so that the reader may at least have a suspicion that the Theory of Numbers is a real science, and not a collection of isolated and merely curious theorems." The introductory matter deals with the definitions and the essential properties of the operations upon integers, and they are handled "independently of the manner in which the operations are effected, and of the decimal numeration." The author feels that logical perfection at this stage is out of place, and runs the risk of seeming a mere matter of words. "The important thing is, not to astonish the students by forms of reasoning which seem to them to be captious, and in which they are more bewildered than convinced, but to let them see that they already possess as part of their mental furniture something of what we are teaching them, to throw further light upon and to exhibit in clear cut form what they have as yet but indistinctly grasped, to lead them little by little to the indispensable minimum of those forms of reasoning which are simple and general because they are abstract." Thus while each chapter deals with the simpler notions it concludes with indications as to the possibility of a more logical method of approach. The problem of the best method of attacking negative number presents itself on p. 30. No appeal is made to geometry, and the author confines himself to concepts arising out of the idea of number. The numbers $+5$, -5, for instance, are said to be symmetrical, and symmetrical numbers are exhibited in the usual way on a line.

$$\ldots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \ldots$$

By convention we consider any positive integer greater than those on its left, and *a fortiori* greater than any negative number. Also, the greater of any two negative numbers is that which has the smaller absolute value. We have found that it is possible to introduce the negative number even to young children by asking them to look into a looking glass placed vertically at 0. The "image" of the 1 in the glass is just as far from the 0 as the 0 is from the 1, the image of the 2 is as far from the 0 as the 0 is from the 2, and so on. To add -3 to 5 is to count three steps towards the glass, *e.g.* 4, 3, 2; to subtract 5 from 3 is to proceed by the five steps: 2, 1, 0, -1, -2, and so on. The process is quite as valid as "la valeur absolue de leur somme est la différence entre . . . et le signe est le signe de celui des deux nombres qui a la plus grande valeur absolue." It has the advantage of giving a certain reality, valuable even if fictitious, to the series of numbers, $-1, -2, -3, \ldots$, of pictorially exhibiting them with reference to the positive or counting integers, and incidentally shows that 0 belongs to both sets. The sections devoted to H.C.F. might be much shortened. It does not take a child long to see that whatever is a factor of 360 and 172 is also a factor of $360 - 2 \times 172$, *i.e.* of 16. Their common factors also are factors of $172 - 10 \times 6$, *i.e.* of 12. The only number that is a factor of 12 and of 16 is 4. Hence, etc. The French method, after the usual pages of explanation which no boy or girl ever understood, is:

	2	10	1	3
360	172	16	12	4
344	160	12	12	
16	12	4	0	

in which the rationale of the method is successfully concealed from start to finish. In the chapter on approximations the author acknowledges his indebtedness to Commandant Guyou's useful *Note sur les approximations numériques*,

which will repay the adventurous purchaser. This chapter is well worth careful study by the young teacher. In the treatment of irrational number, Prof. Tannery adopts Dedekind's fundamental idea, in which the irrational number is defined by stating the rational numbers respectively greater and less than it. A few pages on aggregates and their elements, with upper and lower limits, leads to the general elementary treatment of limits and the conditions for convergence and divergence. Considerations of space forbid further comment. We must content ourselves with the expression of our opinion that few students of methodology will rise from even the casual perusal of this volume without the feeling that their time has not been wasted. Of this, however, the names of Profs. Tannery and Darboux will be in themselves an adequate guarantee.

MATHEMATICAL NOTES.

358. [A. 3. d.] *Note on Sturm's theorem in the theory of equations.*

There is considerable likeness between Sturm's method of testing the reality of the roots of an equation, and the method by which the reality of the roots of the discriminant of $S+\lambda(x^2+y^2+\ldots)$ is proved, S being a quadratic in $x, y, \ldots$; in each case a series of functions is formed such that when one member of the series vanishes its two neighbours have opposite signs.

The object of this note is to point out that the two methods are in essence identical, the fact being that Sturm's process enables us to express any polynomial in x as a symmetrical determinant involving x in the diagonal only.

Let f be the polynomial of the nth degree and $f, f_1, f_2, \ldots f_m$ the series of Sturm's functions, $q_1, q_2, \ldots q_m$ the quotients, so that

$$\begin{aligned} f &= q_1 f_1 - f_2, \\ f_1 &= q_2 f_2 - f_3, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ f_{m-1} &= q_m f_m. \end{aligned}$$

Eliminating $f_1, f_2, \ldots f_{m-1}$, we have

$$f = f_m \begin{vmatrix} q_1, & -1 & & & & \\ -1, & q_2, & -1 & & & \\ & -1, & q_3, & -1 & & \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \\ & & & -1, & q_{m-1}, & -1 \\ & & & & -1, & q_m. \end{vmatrix}$$

In general $m=n$, $q_1, q_2, \ldots$ are linear in x and f_m is a constant, so that f is expressed as a symmetrical determinant, involving x linearly in the diagonal only.

The successive principal minors $A_{11}, A_{1122}, \ldots$ are here $f_1, f_2, f_3, \ldots$, and the identity

$$\Delta_x A_{1122} = A_{11} A_{22} - A_{12}^2$$

of the symmetrical determinant theory becomes

$$f \times f_2 = f_1(q_1 f_2) - f_2^2.$$

The roots of f are all real if the coefficients of x in $q_1, q_2, \ldots$ are all of one sign. In the case of q_1 this sign must be positive, and we have the usual rule that all the roots are real when the leading coefficients in Sturm's functions are all of the same sign.

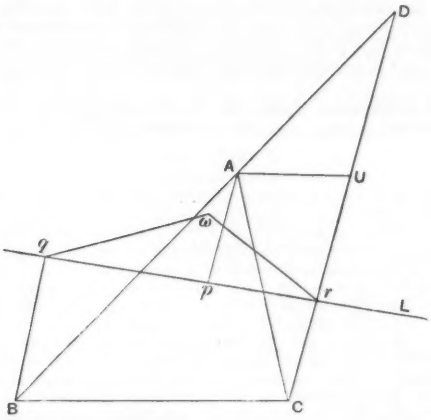
To reduce f to a symmetrical determinant in this way we may take any $(n-1)^{ic}$ for f_1, but for the purpose of placing the roots it is essential that the roots of f_1 should separate those of f; then the roots of each principal minor are separated by those of the next in succession.

One exceptional case is that in which f has one or more repeated roots: f_m is not a constant, but it can be expressed as a symmetrical determinant, and thus f can be so expressed, but zero will occur in the lines next to the diagonal.

The series of Sturm functions may also be shortened through there being a drop of more than one in degree from one function to the next. In this case one of the quotients q_1, q_2, ... will not be linear; f must have imaginary roots. A. C. DIXON.

359. [K. 1; L¹, 13. b.] With reference to Mr. W. Gallatly's note on the "Orthopole" in the *Gazette* for May last (Vol. VI., No. 92, p. 89) the following simple proof of its characteristic property (1) may be noticed:

Let Ap, Bq, Cr be $\perp$rs on a given line L. Draw $q\omega$, $r\omega$ $\perp$rs to AC, AB respectively meeting in ω. It is required to shew that $p\omega$ is $\perp$r to BC. Produce BA, Cr to meet in D, and draw AU parallel to BC to meet DC in U. Then the triangle ACD is similar to the triangle ωqr, corresponding sides being at right angles. Also, since $CU : UD = BA : AD = qp : pr$, therefore U, p are corresponding points on the bases CD, qr. Therefore $p\omega$ is $\perp$r to AU, or to BC.



Under what circumstances is it possible to describe a parabola having L for its directrix in such a manner that ABC shall be a self-conjugate triangle?

It is a well-known property that the joins of the middle points of such a triangle are tangents; hence the orthocentre of the medial triangle (the circumcentre of ABC) must lie on L, that is to say, L must be a circum-diameter. Since Ap, Bq, Cr are then diameters of the parabola, the polars of the points p, q, r on the directrix L are parallel to the sides. Hence the perpendiculars through p, q, r to the respective sides cointersect in the focus ω, which lies on the circumcircle of the medial triangle, which is the N.P. circle of ABC. R. F. DAVIS.

360. [I. 18. c.] On p. 158 of the *Gazette* a question of priority is raised as to the discovery of the identities

$$4^4+6^4+8^4+9^4+14^4=15^4 \text{ and } 4^5+5^5+6^5+7^5+9^5+11^5=12^5,$$

the former being due to Mr. David S. Hart, and the latter to Mr. Artemas Martin, the signatory of the note.

It is a matter of satisfaction to all concerned that the question of priority should thus be placed beyond a doubt. But at the same time it is rather amusing to find that Mr. Artemas Martin also desires to place on record the fact that I have not myself discovered other identities to the authorship of which he lays claim. Needless to say I have nowhere claimed to have discovered everything. After pointing out a general method for the solution of the equation $\sum_1^n x_n^p = y^p$, I limited myself to the solution, the simplest, for $p=4$, a solution, also the simplest, for $p=5$, and added "and so on" (pp. 143, 146 of the monograph under review). So that the field was left free for any investigator who is undeterred by the laborious nature of research of the kind.

Of course the discovery of the two identities given above constitutes but a very small portion of my memoir, which contains a considerable amount of much more interesting material.

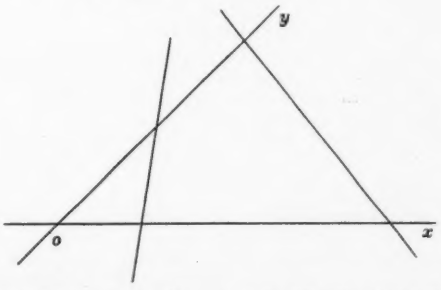
My monograph, completed in 1909, was not printed until 1910, and copies were sent to various learned Societies across the Atlantic. I must add that until the appearance of the note in the *Gazette* I was totally unaware that Mr. Artemas Martin had ever been engaged in researches analogous to my own. It is perhaps unfortunate that the general body of mathematical literature is assuming such proportions that only too few of the mathematicians of the Old World are able to keep themselves *au courant* with what has been or is being published in the New. E. BARBETTE.

Liége, Oct. 1911.

361. [L¹. 14. a.] *On the conics passing through four concyclic points.*

Take two opposite sides of the quadrilateral formed by the points as axes of coordinates, and let the intercepts made by the other two opposite sides be a, b; a', b'. Then the equation of any conic through the four points is

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)\left(\frac{x}{a'}+\frac{y}{b'}-1\right)=\lambda xy.$$



Now the points are concyclic; $\therefore\ aa'=bb'\equiv k^2$; but also $a+a'\equiv 2\alpha$, $b+b'\equiv 2\beta$. Hence the various conics can be tabulated thus:

I. *Circle.* $x^2+2xy\cos\omega+y^2-2\alpha x-2\beta y+k^2=0.$

II. *Parabolas.* $(x\pm y)^2-2\alpha x-2\beta y+k^2=0.$

III. *Rect. hyp.* $x^2+2xy\sec\omega+y^2-2\alpha x-2\beta y+k^2=0.$

I. The circle may be written $(x-x')^2+2(x-x')(y-y')\cos\omega+(y-y')^2=r^2$, where (x', y') is the centre, r the radius.

It is easily found that

$$r^2=(\alpha^2-2\alpha\beta\cos\omega+\beta^2)\operatorname{cosec}^2\omega-k^2.$$

Lemma. The lengths of the perpendicular from the pt. (x', y') on the line $x \pm y - p = 0$ are $(x' \pm y' - p)\dfrac{\cos\frac{\omega}{2}}{\sin\frac{\omega}{2}}$.

II. Consider the parabolas $(x \pm y)^2 - 2\alpha x - 2\beta y + k^2 = 0$.

In the usual way we find that:

the equations of the axes are $2(x \pm y) = \alpha \pm \beta)$,

the equations to the tangents at the vertices are

$$(\alpha \mp \beta)(x \mp y) = k^2 - \left(\frac{\alpha \pm \beta}{2}\right)^2,$$

and the lengths of the latera recta are

$$(\alpha - \beta)\cos\frac{\omega}{2}\cot\frac{\omega}{2} \quad \text{and} \quad (\alpha + \beta)\sin\frac{\omega}{2}\tan\frac{\omega}{2}.$$

The rectangular hyperbola $x^2 + 2xy\sec\omega + y^2 - 2\alpha x - 2\beta y + k^2 = 0$ may be written

$$\left(x + y - \frac{\alpha+\beta}{2}\cdot\frac{\cos\omega}{\cos^2\frac{\omega}{2}}\right)^2\cos^2\frac{\omega}{2} - \left(x - y + \frac{\alpha-\beta}{2}\cdot\frac{\cos\omega}{\cos^2\frac{\omega}{2}}\right)^2\sin^2\frac{\omega}{2}$$
$$= \cot^2\omega\{2\alpha\beta - (\alpha^2 + \beta^2\cos\omega\} - k^2\cos\omega.$$

Hence the equations of the axes are

$$x + y - \frac{\alpha+\beta}{2}\left(1 - \tan^2\frac{\omega}{2}\right) = 0,$$

$$x - y + \frac{\alpha-\beta}{2}\left(\cot^2\frac{\omega}{2} - 1\right) = 0,$$

and the square of the semi-axis is

$$\cot^2\omega\{2\alpha\beta - (\alpha^2 + \beta^2)\cos\omega\} - k^2\cos\omega. \qquad \text{N. M. Gibbins.}$$

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